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Coupling between stationary Marangoni and Cowley-Rosensweig instabilities in a deformable ferrofluid layer

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Abstract: A horizontal thin layer of ferrofluid is bordered by a solid and open to an inert gas on the other side. It is submitted to a heat gradient and a weak magnetic field, both being normal to the free deformable surface, leading to a coupling between the Marangoni phenomenon, induced by the variation of surface tension along the free deformable surface and the isothermal Cowley-Rosensweig problem, consequence of the magnetic field. The study of the steady compatibility condition shows a new pattern of stationary instability. The critical wavenumber is $O(\sqrt{\mathsf{Bo}})$, the Bond number Bo being smaller than 1, at a minima of the Marangoni number, that could be much less thus than its classical undeformable counterpart. For large wavelengthes, the Marangoni number depends on the Galileo number in contradistinction to earlier results.

1 Introduction

A thin layer of ferrofluid is sandwiched between a solid surface and an inert gas, submitted to the joint action of a weak magnetic field and of a gradient of temperature, both normal to the unperturbed horizontal borders of infinite extent. Such a shallow pond enables to neglect all *bulk* forces fluctuations, whether buoyancy or of magnetic origin. The free surface of the ferrofluid layer couples the Marangoni instability due to surface traction along the interface [Pearson (1958)] to the static isothermal Cowley-Rosensweig instability [Rosensweig (1997)], due to the inbalance between the magnetic traction, the surface tension and gravity leading to a change of the shape of the free surface.

In this note, we develop the study of the linear marginal non ocillating coupling between both instabilities [Rosensweig (1997); Bashtovoi and Pavlinov (1979); Pavlinov (1979); Bashtovoi, Berkovski, and Vislovitch (1988); Salin (1993); Hennenberg, Weyssow, Slavtchev, and Legros (2001); Weilepp and Brand (1996)], when the ferrofluid deformable layer rests on the solid wall, or hangs down from it [Smith (1966); Takashima (1981); Velarde, Nepomnyaschy, and Hennenberg (2000)]. Our analysis show that when both isothermal situations (Rayleigh-Taylor and Cowley-Rosensweig) are stable, the Marangoni stability criterion can be modified to give a critical value of the Marangoni number less than the one of Pearson [Pearson (1958)] for a wavelength of the order of the capillary length. Also, we correct the result derived by Bashtovoi and Pavlinov [Bashtovoi and Pavlinov (1979); Pavlinov (1979); Bashtovoi, Berkovski, and Vislovitch (1988)] for the long wavelength approximation which failed to get back the classical results in the absence of a magnetic field [Smith (1966); Takashima (1981); Velarde, Nepomnyaschy, and Hennenberg (2000)]. A complete study is in progress.

2 The Problem

A horizontal layer of a ferrofluid of width d and of infinite lateral extent, is bordered by a nonmagnetic solid (superscript s), located at $z^* = 0$ and by a free limiting surface Σ, that is an infinite flat plane at $z^* = d$ in the reference rest state, which is in contact with a gaseous magnetically inert phase (superscript g). This layer is submitted to a gradient of temperature and to an exterior weak magnetic field, both normal to the unperturbed liquid-gas and liquid-solid interfaces (see Fig. 1).

Ferrofluid magnetic properties

The magnetic field derives from a gradient in all three phases $\mathbf{H}^l = \nabla\phi^l, l = g, s$ and $\mathbf{H} = \nabla\phi$ in the ferrofluid layer, where also the Maxwell equation $\nabla\cdot(\mu_0\,[\mathbf{H}+\mathbf{M}]) = 0$ intervenes, μ_0 being the magnetic void permeability. The magnetic field $\mathbf{H}$ and the ferrofluid magnetisation $\mathbf{M} = \chi\mathbf{H}$ are collinear defining the per-

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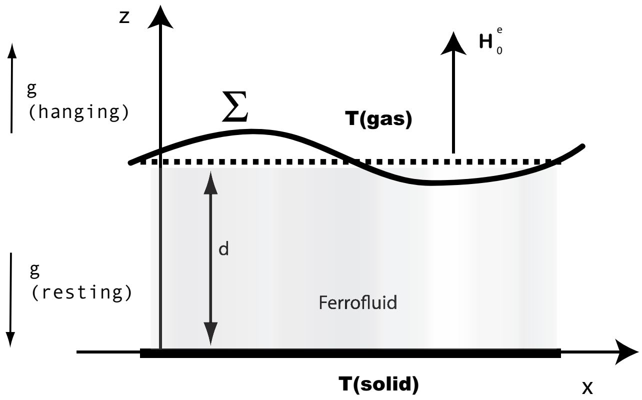

Figure 1 : Ferrofluid layer submitted to a normal constant magnetic field $\mathbf{H}_0^e = H_0^e \mathbf{1}_z$ and to a normal temperature gradient $\Delta T = T(gas) - T(solid)$. $\mathbf{1}_z$ = unit normal directed from solid into gas, $\mathbf{1}_x$ = horizontal unit vector along $z = 0$.

mittivity χ [Bashtovoi and Pavlinov (1979); Pavlinov (1979); Bashtovoi, Berkovski, and Vislovitch (1988); Weilepp and Brand (1996)], whose change with temperature accross the layer is neglected. Then, the Maxwell equations reduce to

$$\nabla^2 \phi^g = 0 \quad \text{for} \quad z \geq d + \xi \quad ,$$
$$\nabla^2 \phi^s = 0 \quad \text{for} \quad z \leq 0 \quad , \tag{1}$$
$$\nabla^2 \phi = 0 \quad \text{for} \quad 0 \leq z \leq d + \xi$$

where $d + \xi$ is the height of the liquid-gas surface Σ. On the upper and lower boundaries of the ferrofluid layer, one has the continuity of the normal components of $\mu_0 [\mathbf{H} + \mathbf{M}]$ and of the tangential component of the magnetic field $\mathbf{H}$ [Rosensweig (1997)].

Balance of momentum and Laplace-Marangoni boundary condition

As a consequence of Eq. 1, for an incompressible viscous ferrofluid, whose constant density is ρ (thus whose specific volume $\mathcal{V} = \rho^{-1}$), the momentum balance law reads:

$$\rho D_t \mathbf{v} = -\nabla p + \eta \nabla^2 \mathbf{v} + \rho \mathbf{g} \quad \text{and} \quad \nabla \cdot \mathbf{v} = 0 \tag{2}$$

where $\mathbf{v} = (U, V, W)$ is the velocity, $p = p_L(\rho, T) + \mu_0 \int_0^H \frac{\partial M \mathcal{V}}{\partial \mathcal{V}}\Big|_{H,T} d\mathcal{V}$ is the total pressure, $p_L(\rho, T)$ is the hydrostatic pressure, D_t is the operator $\partial_t + \mathbf{v} \cdot \nabla$, and η is the kinematic viscosity. Since we are supposing that $\mathbf{1}_z$ is always directed from the solid boundary at $z = 0$ toward the deformable surface Σ at $z = d + \xi$, two cases are summarized by the gravity field $\mathbf{g} = -g \mathbf{1}_z$. If $g = |g|$, we are considering a ferrofluid resting *above* a solid non magnetic border. When $g = -|g|$, the magnetisable layer is hanging *below* the solid ceiling. This extends Rayleigh-Taylor instability to a magnetized ferrofluid submitted to a vertical gradient of temperature [Chandrasekhar (1981); Burgess, Juel, Cornick, Swift, and Swinney (2001); Pacitto, Filament, Bacri, and Widom (2000)]. The boundary conditions on momentum on the solid-liquid interface are $\mathbf{v} = 0$ or $U = V = W = 0$ at $z = 0$. The deformable liquid-gas interface Σ is defined by the Monge equation $\mathbf{r} = x \mathbf{1}_x + y \mathbf{1}_y + \xi(x, y, t) \mathbf{1}_z$ so that the unit normal linearised expression is $\mathbf{n} = -\partial_x \xi \mathbf{1}_x - \partial_y \xi \mathbf{1}_y + \mathbf{1}_z$ [Hennenberg, Weyssow, Slavtchev, and Legros (2001)].

Let us call $\left[T_{ij}^l - T_{ij}^g\right]\Big|_\Sigma n_j = \mathcal{F}_i$, the projection on the normal $\mathbf{n}$ at the interface Σ of the difference between T_{ij}^l the stress tensor in the liquid phase and T_{ij}^g the stress tensor in the inviscid magnetically inert gaseous phase. Then along Σ, one has the following linerarized Marangoni-Laplace condition [Hennenberg, Weyssow, Slavtchev, and Legros (2001); Weilepp and Brand (1996)]:

$$\mathcal{F}_i = 2\mathcal{K} \sigma \delta_{iz} + (1 - \delta_{iz}) \frac{\partial \sigma}{\partial x_i} \tag{3}$$

where T_{ij}^l and T_{ij}^g are respectively

$$\begin{aligned} T_{ij}^l &= -\left\{p + \frac{\mu_0}{2} H^2\right\} \delta_{ij} \\ &\quad + \mu_0 (1 + \chi) H_i H_j + \eta \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right] \\ T_{ij}^g &= -\left\{p_{gas} + \frac{\mu_0}{2} H^2\right\} \delta_{ij} + \mu_0 H_i H_j \end{aligned} \tag{4}$$

where δ_{ij} is the Kronecker delta. The surface mean curvature is $2\mathcal{K} = \partial_x^2 \xi + \partial_y^2 \xi$, [Hennenberg, Weyssow, Slavtchev, and Legros (2001)]. Since gas and liquid are immiscible, $D_t \xi_s = \mathbf{v}|_\Sigma \cdot \mathbf{1}_z$.

Heat balance, state equation and boundary conditions

The energy equation reduces to the usual Fourier equation [Rosensweig (1997)]:

$$\rho c_{p,H} D_t T = \lambda \nabla^2 T \tag{5}$$

where $c_{p,H}$ is the specific heat capacity at constant pressure and magnetic field, λ is the thermal conductivity. Along the free deformable liquid-gas surface Σ, the heat flux will be proportional to the difference between the surface temperature and the temperature T_{gas} of the gaseous phase:

$$-\lambda\left[\mathbf{n}\cdot\nabla T\right]\big|_{\Sigma} = a\left[T\big|_{\Sigma} - T_{gas}\right] \tag{6}$$

where a is the heat transfer coefficient. The surface tension varies linearly with temperature, so that $\sigma = \sigma_0\left[1-\gamma(T-T^0_{lg})\right]$ where T^0_{lg} is the reference liquid gas temperature, σ_0 is the value of the surface tension at T^0_{lg} and $\gamma = -\dfrac{1}{\sigma_0}\dfrac{\partial\sigma}{\partial T}$ is a positive quantity. Along the other boundary, the solid is a perfect conductor, so that $T = T\big|_{wall} = Const$ at $z = 0$. The reference temperature at the lower solid-liquid surface will hereafter be denoted T_{sol}.

The reference rest state

The steady solution of Eq. 5 is:

$$T^0 = T_{sol} - \beta z \tag{7}$$

A conducting liquid-gas interface corresponds to the case $a\to\infty$, and an insulating one to $\lambda\to\infty$. The quantity $\beta = a\left[T_{sol}-T_{gas}\right]/(a\,d+\lambda)$ depends on which boundary interface is the heating one, so that β is *positive* when heating from the *solid wall* ($T_{sol} > T_{gas}$) and *negative* when heating from the *gaseous phase* ($T_{sol} < T_{gas}$).

The ferrofluid is submitted to an exterior constant magnetic field $\mathbf{H} = \mathbf{1}_z H^e_0$. Thus, the Maxwell equations Eq. 1 give the unperturbed magnetic field H_0 and the unperturbed magnetisation M_0 in the ferrofluid layer as:

$$H^e_0 = H_0 + M_0 = (1+\chi)\,H_0$$

The continuity of the normal component of the induction and of the stresses, accross the reference liquid gas interface leads to the well known magnetic pressure jump [Rosensweig (1997); Weilepp and Brand (1996)]

$$p_{gas} - p_{liq} = \frac{\mu_0}{2}\left[\chi H_0\right]^2 = \frac{\mu_0}{2}M_0^2 \tag{8}$$

3 The dimensionless linear perturbation of the state

To study the linear stability of the reference motionless conductive state Eq. 7 - Eq. 8, we write the problem in a dimensionless form. We use the following scaling units [Bashtovoi and Pavlinov (1979); Pavlinov (1979); Bashtovoi, Berkovski, and Vislovitch (1988); Weilepp and Brand (1996)]: any spatial dimension is scaled by d (so that the reference free surface is $z = 1$), the time by d^2/κ ($\kappa = (c_{p,H}\,\rho)^{-1}$ is the thermal diffusivity), the temperature by βd and the magnetic field or the magnetization by $M_0/(1+\chi) = \chi H_0/(1+\chi) = \chi H^e_0/(1+\chi)^2$. Each dimensionless perturbed quantity δf keeps its former symbol f to identify its physical origin and we develop it in a Fourier expansion in normal modes so that we must keep only one single mode in the form $\delta f(z)\exp[\imath\,(k_x x + k_y y) + \omega t]$ [Bashtovoi and Pavlinov (1979); Weilepp and Brand (1996); Chandrasekhar (1981)]. The dimensionless wavenumber $\mathbf{k} = (k_x, k_y)$ has real components and $\omega = \Re(\omega) + \imath\,\Im(\omega)$, where $\Re(\omega)$ shows whether the situation is stable ($\Re(\omega) < 0$), marginally stable ($\Re(\omega) = 0$) or unstable ($\Re(\omega) > 0$), while $\Im(\omega)$ being different from zero indicates an oscillating solution. Calling D, the differential operator $D = d/dz$ and introducing $k = \sqrt{k_x^2 + k_y^2}$, we obtain the normal mode dimensionless formulation of the problem. We restrict our analysis to the non oscillating marginal case $\Re(\omega) = \Im(\omega) = 0$.

The dimensionless Maxwell equation Eq. 1 give us

$$\begin{aligned}
\left[D^2 - k^2\right]\delta\phi &= 0 \quad \text{for} \quad 0 \le z \le 1+\xi \\
\left[D^2 - k^2\right]\delta\phi^g &= 0 \quad \text{for} \quad z \ge 1+\xi \\
\left[D^2 - k^2\right]\delta\phi^s &= 0 \quad \text{for} \quad z \le 0
\end{aligned} \tag{9}$$

The momentum balance Eq. 2 describing the ferrofluid layer becomes

$$\left(D^2 - k^2\right)^2 W = 0 \tag{10}$$

The energy equation Eq. 5 leads to its dimensionless equivalent

$$\left(D^2 - k^2\right)\delta T + W = 0 \tag{11}$$

The physical relevance of Eq. 10 and Eq. 11 assumes to study only cases where $d << 1/\sqrt{\rho g/\sigma}$ [Velarde, Nepomnyaschy, and Hennenberg (2000)].

Boundary conditions at the deformable surface Σ

For any scalar quantity g and for any vector $\mathbf{f}$ taken along the deformed surface Σ, their linear perturbation is defined as the sum of two contributions [Bashtovoi and Pavlinov (1979); Pavlinov (1979); Hennenberg, Weyssow, Slavtchev, and Legros (2001)]

$$\delta g_\Sigma = \delta g_1 + \frac{\partial g}{\partial z}\xi \quad \text{and} \quad \delta \mathbf{f}_\Sigma = \delta \mathbf{f}_1 + \mathbf{n}\cdot\nabla \mathbf{f}$$

Introducing the Biot number $\mathsf{Bi} = a\,d/\lambda$, the dimensionless expression of Eq. 6, along the deformed surface Σ for which $W = 0$, is

$$D\,\delta T = -\,\mathsf{Bi}\,[\delta T - \xi] \tag{12}$$

The dimensionless lateral component of Eq. 3 is independent upon the presence of a magnetic field and is the usual Marangoni tangential shear stress balance:

$$\left[D^2 + k^2\right] W + \mathsf{Ma}\,k^2\,[\delta T - \xi] = 0 \tag{13}$$

with $\mathsf{Ma} = -\dfrac{\partial\sigma}{\partial T}\dfrac{\beta d^2}{\eta\kappa}$ being the Marangoni number.

The Maxwell boundary conditions on the liquid-gas surface give the following dimensionless result:

$$\xi = \frac{\delta\phi - \delta\phi^G}{1+\chi}, \quad \text{and} \quad D\,\delta\phi = k\left[\xi - \frac{\delta\phi}{1+\chi}\right]$$

Introducing the following dimensionless numbers - the crispation number $\mathsf{Cr} = \mu\kappa/\sigma d$, the Bond number $\mathsf{Bo} = \rho g d^2/\sigma$ with $\sqrt{\mathsf{Bo}} < 1$, the magnetic Bond number $\mathsf{Bo}_m = \mu_0\,(\chi H_0^l)^2\,d/\sigma(1+\chi)$, the Galileo number $\mathsf{Ga} = g d^3/\nu\kappa = \mathsf{Bo}/\mathsf{Cr}$ [Velarde, Nepomnyaschy, and Hennenberg (2000); Abou, de Surgy, and Wesfreid (1997)] - enables us to obtain the final dimensionless expression of the Laplace equation derived from Eq. 3 and Eq. 4:

$$k^2\Delta^\pm\xi + \frac{1}{\mathsf{Ga}}\left[3k^2 - D^2\right] DW + k^3\,\frac{\mathsf{Bo}_m}{\mathsf{Bo}}\,\delta\phi = 0 \tag{14}$$

where $\Delta^\pm = +\dfrac{k^2}{\mathsf{Bo}} - k\dfrac{\mu_0\,(\chi H_0^l)^2}{\rho g d} \pm 1$. In $\Delta^\pm$, the upperscipt $+$ (respectively $-$) means a ferrofluid layer resting on the underneath rigid wall (ferrofluid layer hanging down from the upper rigid wall) which corresponds to the $+$ ($-$) sign in front of 1. The magnetic Bond number Bo_m is due to the magnetic pressure jump along the free surface [Rosensweig (1997); Hennenberg, Weyssow, Slavtchev, and Legros (2001); Abou, de Surgy, and Wesfreid (1997); Bacri, Perzynski, and Salin (1988)].

We will suppose the solid wall to be a perfect heat conductor, so that, we have at $z = 0$,

$$W = DW = \delta T = 0 \quad \text{and}$$
$$\delta\phi^S = \delta\phi \quad \text{so that} \quad D\,\delta\phi - \frac{k\,\delta\phi}{1+\chi} = 0 \tag{15}$$

From Eq. 9 and using the boundary conditions Eq. 15 at the wall, the magnetic potential reads [Weilepp and Brand (1996)] along $z = 1$:

$$\delta\phi(1) = \xi\,(1+\chi)\,\Lambda(k) \tag{16}$$

where $\Lambda(k) = (\mu\tanh k + 1)/(\left[\mu^2+1\right]\tanh k + 2\mu)$. The function $\Lambda(k)$ is a monotoneous increasing function from its minimum value $1/2\mu$ at $k = 0$ up to its maximum $1/(1+\mu)$ at $k = \infty$ since the relative permeability $\mu = 1+\chi$ is always larger than one [Rosensweig (1997); Weilepp and Brand (1996)].

From Eq. 12, Eq. 13, Eq. 14, using Eq. 16, we obtain the following compatibility condition that takes into account the Rayleigh-Taylor case:

$$\mathsf{Ma} = \mathsf{Ma}^\pm(k) = 8k\,\times$$
$$\times\frac{[\cosh k\sinh k - k]\,[k\sinh k + \mathsf{Bi}\cosh k]}{\left(\sinh^3 k - k^3\cosh k\right) + \dfrac{8\,\mathsf{Cr}\,k^5\,\cosh k}{\pm\mathsf{Bo} + k^2 - k\Lambda(k)\,\mathcal{N}_m}} \tag{17}$$

where by definition $\mathcal{N}_m = [1+\chi]^2\,\mathsf{Bo}_m$ is directly linked to the magnetic Bond number. When the magnetic field is absent $\mathcal{N}_m = 0$, we find back from Eq. 17 the Marangoni problem studied by Smith and Takashima [Smith (1966); Takashima (1981)]. The term multiplying Cr couples the classical Marangoni case studied from Pearson onwards [Pearson (1958); Smith (1966); Takashima (1981); Velarde, Nepomnyaschy, and Hennenberg (2000)] and the isothermal Cowley-Rosensweig instability [Rosensweig (1997); Hennenberg, Weyssow, Slavtchev, and Legros (2001); Abou, de Surgy, and Wesfreid (1997); Bacri, Perzynski, and Salin (1988)].

a) Indeed, should we neglect the deformation and thus use $Cr = 0$, we obtain $\mathsf{Ma} = \mathsf{Ma}_0(k)$ where

$$\mathsf{Ma}_0(k) = 8k\,\times$$
$$\times\frac{(k - \cosh k\sinh k)\,(k\sinh k + \mathsf{Bi}\cosh k)}{k^3\cosh k - \sinh^3 k} \tag{18}$$

This is the classical Marangoni compatibility condition [Pearson (1958)], whose critical value is $\mathsf{Ma}_0(1.992) \approx 79.6$, at a critical wavenumber $k_{crit} \approx 1.992$.

b) If the fluid is isothermal, $\mathsf{Ma} = 0$. But since the numerator of Eq. 17 is always positive, the compatibility condition Eq. 17 reduces to zero only if the denominator is infinite, which means to have $\Delta_d^{\pm}(k) = 0$, where we define

$$\Delta_d^{\pm}(k) = +\frac{k^2}{\mathsf{Bo}} - \frac{k\Lambda(k)\,\mathcal{N}_m}{\mathsf{Bo}} \pm 1 \qquad (19)$$

But $\Delta_d^{\pm}(k) = 0$ corresponds to the generalisation of the compatibility condition of the Cowley-Rosensweig instability for any layer width d [Rosensweig (1997); Abou, de Surgy, and Wesfreid (1997); Bacri, Perzynski, and Salin (1988); Chandrasekhar (1981)]. If we use $K = k/\sqrt{\mathsf{Bo}}$ and the function $\Phi = \mathcal{N}_m/2\,(1+\mu)\sqrt{\mathsf{Bo}}$ [Abou, de Surgy, and Wesfreid (1997)], we can rewrite last equation Eq. 19 as

$$\Delta_\infty^{\pm}(K) \le \Delta_d^{\pm} \le \Delta_\infty^{\pm}(K) + K\Phi\,\frac{(\mu-1)}{\mu} \qquad (20)$$

where $\Delta_\infty^{\pm}(K) = K^2 - 2K\Phi \pm 1$, $\Delta_d^{\pm} = \Delta_\infty^{\pm}(K) - \alpha\Phi$, and $\alpha = (1+\mu)\,\frac{\mu\tanh(K\sqrt{\mathsf{Bo}})+1}{(\mu^2+1)\tanh(K\sqrt{\mathsf{Bo}})+2\mu} - 2K$. The above inequality defines two extreme cases. A thick layer supposes the width d to be much larger than the capillary length $\sqrt{\sigma/\rho g}$ [Rosensweig (1997); Salin (1993); Hennenberg, Weyssow, Slavtchev, and Legros (2001); Abou, de Surgy, and Wesfreid (1997)]. The isothermal inviscid Cowley-Rosensweig instability reduces to the study of $\Delta_\infty^{\pm}(K\sqrt{\mathsf{Bo}}) = 0$ where $\Delta_\infty^{\pm}(K\sqrt{\mathsf{Bo}}) = K^2 - 2K\Phi \pm 1$. A very very thin layer exists when the capillary length is much more larger than the width d [Abou, de Surgy, and Wesfreid (1997); Bacri, Perzynski, and Salin (1988)] and the compatibility condition is the study of $\Delta_0^{\pm}(K\sqrt{\mathsf{Bo}}) = 0$, where $\Delta_0^{\pm}(K\sqrt{\mathsf{Bo}}) = \Delta_\infty^{\pm}(K) + K\Phi\mu^{-1}(1+\mu)$.

4 Preliminary results

Using Eq. 18 and Eq. 19 we can rewrite Eq. 17 as

$$\mathsf{Ma}^{\pm}(k) = \mathsf{Ma}_0(k)\,\mathcal{A} \qquad (21)$$

where $\mathcal{A} = \Delta_d^{\pm}(k)/\left\{\Delta_d^{\pm}(k) + \frac{8k^5\cosh k}{Ga\left[\sinh^3 k - k^3\cosh k\right]}\right\}$. We will restrict ourselves to some preliminary results and discuss the longwavelength approximation of Eq. 21.

4.1 The ferrofluid layer resting on a solid surface

Increasing the magnetic field, Eq. 20 shows that $\Delta_d^{+}(k)$ has either no positive real root (Cowley-Rosensweig stable case), one positive zero (Cowley-Rosensweig marginal stability) or two positive roots (Cowley-Rosensweig unstable case) [Rosensweig (1997); Hennenberg, Weyssow, Slavtchev, and Legros (2001); Salin (1993); Abou, de Surgy, and Wesfreid (1997)].

A] $\Delta_d^{+}(k)$ has less than two positive roots

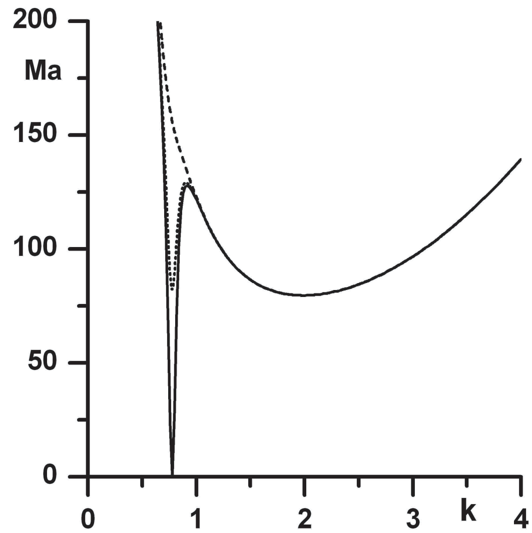

Figure 2 : Neutral curves $\mathsf{Ma}^{+}(k)$ for $k > 1$ (Pearson curve). (a) Cowley-Rosensweig marginal case (solid line), (b) Cowley-Rosensweig stable case showing a finite minima at $k \sim O(\sqrt{\mathsf{Bo}})$ (dotted line), and (c) case when the magnetic field has a negligible influence (dashed line).

When $\Delta_d^{+}(k)$ is non negative, the RHS of Eq. 21 is the product of the function $\mathsf{Ma}_0(k)$ by a non negative function $\mathcal{A}$ less than one. For very large values of Ga, $\mathcal{A} \approx 1$ the interface is practically undeformable so that this explains why the dashed curve in Fig. 2 followed by the solid line gives back the classical curve of Pearson [Pearson (1958)]. However for lower value of Ga, where the interface can deform, $\mathcal{A}$ is less than 1 so that it will decrease the value of $\mathsf{Ma}^{+}(k)$ with respect to $\mathsf{Ma}_0(k)$, leading to a new minimum of the curve. One could obtain a new critical wavelength giving rise to the same value of the critical Marangoni number for two different critical values of the wavenumber (dotted curve in Fig. 2). Increasing still the magnetic field up to its marginal value, one observes that this new critical wavenumber becomes the leading one. The Cowley-Rosensweig isothermal problem is stable but the coupling allows a lower gradient of temperature to reach the marginal Marangoni value at a critical wavelength still of $O(d/\sqrt{\mathsf{Bo}})$. This exists until $\mathcal{A} = 0$, where the overall Marangoni problem $\mathsf{Ma}^{+}(k)$ is equal to zero at that finite wavenumber $k_{crit} \approx O(\sqrt{\mathsf{Bo}})$, since the critical wavenumber of the isothermal Cowley-Rosensweig instability is $\sqrt{\mathsf{Bo}}$, both for infinitely thin $\Delta_0^+(k) = 0$ and large layer $\Delta_\infty^+(k) = 0$ [Rosensweig (1997); Abou, de Surgy, and Wesfreid (1997); Salin (1993); Bacri, Perzynski, and Salin (1988); Hennenberg, Weyssow, Slavtchev, and Legros (2001)]. The RHS of Eq. 21 is positive for all other wavenumbers. The stable Cowley and Rosensweig magnetic field induces a new possible Marangoni pattern when *heating from below* (Fig. 2). For the non oscillating case, heating from above, is physically meaningless since the RHS of Eq. 21 is non negative.

B] $\Delta_d^+(k)$ has two different positive roots

If $\Delta_d^+(k)$ has two roots k_- and k_+, the isothermal inviscid Cowley-Rosensweig case is unstable for k such that $k_- \le k \le k_+$, leading to change of shape of the free surface [Rosensweig (1997)]. But for very large Ga, there cannot be any coupling between the Marangoni problem and the Cowley-Rosensweig one, since the gradient of temperature is applied to a completely rigidified surface where the Marangoni problem gives back the result of Pearson [Pearson (1958)].

I] For highly deformable surface where Ga is much smaller, one will have $|\Delta_d^+(k)| << \dfrac{8k^5 \cosh k}{Ga\left[\sinh^3 k - k^3 \cosh k\right]}$ for every k in the interval $[k_-, k_+]$, the numerator of $\mathcal{A}$ and its denominator are of opposite sign, so that $\mathsf{Ma}^{+}(k)$ is negative in the interval $[k_-, k_+]$ and strictly positive outside that interval. Thus whatever the direction of heating and the heat jump, the coupling leads to an unstable Marangoni problem.

II] In an intermediary range of Ga, the denominator of $\mathcal{A}$ might become equal to zero at wavelengths k_1 and k_2, such that $k_- < k_1 < k_2 < k_+$ so that we will have two singularities since there $|\mathsf{Ma}^{+}(k)|$ becomes infinite. Again, whatever the direction of heating, exists an unstable wavenumber interval.

When the isothermal case is unstable by itself, there exists thus a critical Galileo number such that larger values of it amounts to uncouple both problems. However, for lower values of the Galileo number, the Marangoni problem is always unstable, whatever the heating direction or the applied temperature gradient. The coupling looses thus every interest since it considers a surface whose shape has stopped to exist.

4.2 The ferrofluid layer hanging down from the ceiling

Then $\Delta_d^-(k)$ has always one and only one real positive root $k = k_0$ and is negative fom $k = 0$ up to k_0, where thus the overall Marangoni number Ma^- given by the RHS of Eq. 21 is equal to zero. For wavenumbers larger than k_0, $\Delta_d^-(k)$ is positive so that the RHS of Eq. 21 is positive. Between $k = 0$ and $k = k_0$, the denominator of $\mathcal{A}$ is the sum of $\Delta_d^-(k)$, a negative function monotoneously increasing from -1 at $k = 0$ up to 0, and a positive function $\dfrac{8k^5 \cosh k}{Ga\left[\sinh^3 k - k^3 \cosh k\right]}$ that is equal to zero at $k = 0$ and $k = \infty$. Thus this denominator has always one root $k = k_{|\mathsf{Ma}|=\infty}$ *smaller* than k_0, where the Marangoni number given by the RHS of Eq. 21 becomes singular. The Marangoni number Ma^- is thus positive for $0 \le k \le k_{|\mathsf{Ma}|=\infty}$ since both the numerator and the denominator of $\mathcal{A}$ are negative) and for all wavenumbers larger than k_0 since both the numerator and the denominator of $\mathcal{A}$ are positive. The Marangoni number Ma^- is negative for all wavenumbers k such that $k_\infty < k < k_0$. The problem is always unstable due to the Rayleigh-Taylor instability [Chandrasekhar (1981)], but the magnetic field intervenes to change the critical wavenumber k for which $\Delta_d^{\pm}(k)$ is equal to zero. The isothermal Rayleigh-Taylor instability makes the Marangoni instability unstable, whatever the direction of heating.

4.3 The long wavelength approximation

Since the Galileo number is anyway rather large ([Rosensweig (1997); Weilepp and Brand (1996)]), the fraction multiplying $\mathsf{Ma}_0(k)$ differs from unity by an error that decreases as $O(\dfrac{32k^5}{Ga * \exp 2k})$, with increasing k. For large wavenumbers $k \ge 3$, thus the magnetic field H^e and the deformation have a very minute role. We find back the solution given by Pearson [Pearson (1958)], *independent upon gravity and upon magnetic field* $\mathcal{N}_m$. On the contrary, for *long wavelengthes*, we develop Eq. 17 up to the term multiplying k^2. To do that in a meaningful way, we have however to go to higher order terms in the series development of $\cosh k, \sinh k$ and $\tanh k$. Then, from Eq. 21, we have

$$\lim_{k\to 0} \mathsf{Ma} = \frac{2}{3}\mathsf{Ga}\,(1+\mathsf{Bi})\left\{\Delta_0^{\pm}(k) - k^2\left[\pm\frac{2}{15} + \frac{\mathsf{Ga}}{120} + \frac{N_m}{\mathsf{Bo}}\frac{\mu^2-1}{4\mu^2} \mp \frac{1}{3(1+\mathsf{Bi})}\right]\right\} \tag{22}$$

This expression differs from the result of Bashtovoi and Pavlinov [Bashtovoi and Pavlinov (1979); Pavlinov (1979); Bashtovoi, Berkovski, and Vislovitch (1988)] whose asymptotic formula reads

$$\lim_{k\to 0} \mathsf{Ma} = \frac{2}{3}\mathsf{Ga}\,(1+\mathsf{Bi})\,\Delta_0^{\pm}(k) \tag{23}$$

In our opinion, Bashtovoi and Pavlinov went too far in their long wavelength simplification, neglecting a term $O\left(k^2\left[\frac{\mathsf{Ga}}{120} + \frac{\mathcal{N}_m}{\mathsf{Bo}}\frac{\mu^2-1}{4\mu^2}\right]\right)$ that is of the same order as the one they kept $\Delta_0^{\pm}(k)$, (see for example [Weilepp and Brand (1996)]). Let us note that, in the absence of magnetic field, Eq. 22 gives back the result of Takashima and Smith [Smith (1966); Takashima (1981); Velarde, Nepomnyaschy, and Hennenberg (2000)] obtained for usual Newtonian fluids, which is out of question starting with Eq. 23.

5 Conclusion

A magnetic field, less than its Cowley-Rosensweig marginal value, can be coupled to a gradient of temperature. It will influence the Marangoni instability, for a highly deformable surface, and will affect an interval of wavenumber centered around $O\sqrt{\mathsf{Bo}}$, at a lower value of the temperature gradient. The critical value of the Marangoni number lies well below its classical value for the undeformable surface. Also, we corrected the long wavelength approximation found in the literature.

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References

Abou, R.; de Surgy, G. N.; Wesfreid, J. E. (1997): Dispersion Relation in a Ferrofluid Layer of any Thickness and Viscosity in a normal magnetic Field: Asymptotic Regimes. *J. Phys. II France*, vol. 7, pp. 1150–1171.

Bacri, J. C.; Perzynski, R.; Salin, D. (1988): Instability of a ferrofluid film. *C. R. Acad. Sci. Paris, Série II*, vol. 307, pp. 699–704.

Bashtovoi, V. G.; Berkovski, B. M.; Vislovitch, A. N. (1988): *Introduction to Thermomechanics of Magnetic Fluids*. Hemisphere Publ. Corp., Washington D. C.

Bashtovoi, V. G.; Pavlinov, M. I. (1979): Thermocapillary instability of a layer of magnetic fluid with a thermally insulated boundary. *Magnitnaya Gydrodynamika*, vol. 15, pp. 22–26.

Burgess, J. M.; Juel, A.; Cornick, W. D. M.; Swift, J. B.; Swinney, H. L. (2001): Suppression of Dripping from a ceiling. *Phys. Rev. Lett.*, vol. 86, pp. 1203–1206.

Chandrasekhar, S. (1981): *Hydrodynamic and Hydromagnetic Stability*. Dover Publ.

Hennenberg, M.; Weyssow, B.; Slavtchev, S.; Legros, J. C. (2001): Coupling between Marangoni and Rosensweig Instabilities. *Eur. Phys. J. AP*, vol. 16, pp. 217–229.

Pacitto, G.; Filament, C.; Bacri, J. C.; Widom, M. (2000): Rayleigh-Taylor instability with magnetic fluids: Experiment and theory. *Phys. Rev. E*, vol. 62, pp. 7941–7948.

Pavlinov, M. I. (1979): Convective instability in a layer of magnetizable liquid with a deformable free surface, due to thermocapillary mechanism. In *Features of Heat and Mass Transfer, Proc. of A.B. Luikov Inst. Heat and Mass Transfer (in Russian)*, pp. 63–67. Belarus Acad. Sci., Minsk.

Pearson, J. R. A. (1958): On convection cells induced by surface tension. *J. Fluid Mech.*, vol. 4, pp. 489–500.

Rosensweig, R. E. (1997): *Ferrohydrodynamics*. Dover Publ.

Salin, D. (1993): Wave vector selection in the Instability of an Interface in a Magnetic or Electric Field. *Europhys. Lett.*, vol. 21, pp. 667–670.

Smith, K. A. (1966): On convective instability induced by surface tension gradients. *J. Fluid Mech.*, vol. 24, pp. 401–414.

Takashima, M. (1981): Surface tension driven Instability in a Horizontal Liquid Layer with a Deformable Free Surface. Part I: Stationary Convection. *J. of the Phys. Soc. of Japan*, vol. 50, pp. 2745–2750.

Velarde, M. G.; Nepomnyaschy, A.; Hennenberg, M. (2000): Onset of Oscillatory Interfacial Instability and Wave Motions in Bénard Layers. *Adv. in Appl. Mech.*, vol. 37, pp. 168–238.

Weilepp, J.; Brand, H. R. (1996): Competition between the Bénard-Marangoni and the Rosensweig Instability of Magnetic Fluids. *J. Phys. II France*, vol. 5, pp. 419–441.